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ON STABILITY IN THE FIRST APPROXIMATION†

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In the problem of stability in the first approximation, in the sense of Lyapunov, Poincaré and Zhukovskii, the classical condition for regularity of the first approximation is replaced by the requirement that the sign of the Lyapunov exponents must remain constant for small changes in the initial states.

What happens in the neighbourhood of the solution of a non-linear differential equation if the Lyapunov exponents of its linearization are known? Most classical results in this context relate to investigation of stability in Lyapunov's sense. Lyapunov showed that, if the linear system of the first approximation is regular and all its Lyapunov exponents are negative, the solution under consideration is asymptotically stable [1].

Perron [2] showed that the regularity condition is essential and gave an example of a solution which is unstable in Lyapunov's sense, linearization along which is not regular and which has negative Lyapunov exponents. Chetayev proved an analogous theorem for instability [3, 4]: if the linear system of the first approximation is regular and at least one of its Lyapunov exponents is positive, the solution is unstable in Lyapunov's sense.

When investigating attractors of dynamical systems, one often has to consider certain ensembles of solutions [5–9] rather than individual solutions. Numerical methods are now available, and a large number of computer experiments have been carried out, evaluating the Lyapunov exponents of such ensembles [8–10]. From the standpoint of such an analysis of attractors, it is quite natural to try to replace the regularity condition for the first approximation, which is frequently difficult to verify, by the condition that the sign of the Lyapunov exponents must be preserved under "small changes of the initial states".

It has turned out that this can be done using Theorems 1 and 2, proved below.

It should be noted that, even in the analysis of periodic solutions of autonomous systems, other notions of stability become necessary. The most important among these is Poincaré's concept of orbital stability (referred to henceforth also as Poincaré stability). Here one has the well-known Andronov–Vitt theorem [11, 12] and its extension by Demidovich to the case of non-periodic trajectories [13]. Demidovich showed that, if the linear approximation along a bounded trajectory is regular, one of its Lyapunov exponents is zero and all the others are negative, then the trajectory is orbitally stable. Our Theorem 3, proved below, enables one here, too, to drop the regularity property, replacing it by the condition that the Lyapunov exponents of a certain naturally defined linearization remain negative for small changes in the initial states.

When studying the instability of trajectories on attractors, one has to introduce the notion of instability in Zhukovskii's sense [14]. To clarify the difficulties that arise here, we recall the basic definitions of stability for a system

$$dx/dt = f(x), \quad x \in \mathbf{R}^n, \quad f \in C^2 \tag{1}$$

Definition 1. A solution $x(t, x_0)$ of system (1) with initial data $x(0, x_0) = x_0$ is said to be stable in Lyapunov's sense (henceforth, Lyapunov stable) if, for any number $\varepsilon > 0$, a number $\delta(\varepsilon) > 0$ exists such that, for any vector y_0 satisfying the inequality $|x_0 - y_0| \leqslant \delta(\varepsilon)$ and any $t \geqslant 0$, it is true that $|x(t, x_0) - x(t, y_0)| \leqslant \varepsilon$. If, moreover, for some number δ_0 and all y_0 in the sphere $\{y \mid |y - x_0| \leqslant \delta_0\}$ it is true that

$$\lim_{t \to +\infty} |x(t, x_0) - x(t, y_0)| = 0$$

then the solution $x(t, x_0)$ is said to be asymptotically Lyapunov stable.

Here $|\cdot|$ denotes the Euclidean norm in $\mathbf{R}^n$.

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We now introduce the following notation: $L^+(x_0) = \{x(t, x_0) \mid 0 \leq t < +\infty\}$. Thus, the set $L^+(x_0)$ is a positive semi-trajectory of system (1).

Definition 2. A solution $x(t, x_0)$ of system (1) is said to be stable in Poincaré's sense (or Poincaré stable, or orbitally stable) if, for any number $\varepsilon > 0$, a number $\delta(\varepsilon) > 0$ exists such that, for any vector y_0 satisfying the inequality $|x_0 - y_0| \leq \delta(\varepsilon)$ and for any $t \geq 0$

$$\rho(x(t, y_0), L^+(x_0)) \leq \varepsilon \tag{2}$$

If, moreover, for some number δ_0 and all t_0 in the sphere $\{y \mid y - x_0 \mid \leq \delta_0\}$ it is true that

$$\lim_{t \to +\infty} \rho(x(t, y_0), L^+(x_0)) = 0$$

then the solution $x(t, x_0)$ is said to be asymptotically Poincaré stable (or asymptotically orbitally stable).

Here $\rho(z, L)$ denotes the distance between a point z and a set L

$$\rho(z, L) = \inf_{y \in L} |z - y|$$

To define stability in Zhukovskii's sense, we have to consider the following set of homeomorphisms

$$\mathrm{Hom} = \{\tau(\cdot) \mid \tau: [0, +\infty) \to [0, +\infty), \tau(0) = 0\}$$

The functions $\tau(t)$ of the set Hom will play the part of time reparametrizations for the trajectories of system (1).

Definition 3. A solution $x(t, x_0)$ of system (1) is said to be stable in Zhukovskii's sense (or Zhukovskii stable) if, for any number $\varepsilon > 0$, a number $\delta(\varepsilon) > 0$ exists such that, for any vector y_0 satisfying the inequality $|x_0 - y_0| \leq \delta(\varepsilon)$, a function $\tau(\cdot) \in \mathrm{Hom}$ exists for which it is true that $|x(t, x_0) - x(\tau(t), y_0)| \leq \varepsilon, \forall t \geq 0$. If, moreover, for some number $\delta_0 > 0$ and any y_0 in the sphere $\{y \mid x_0 - y_0 \mid \leq \delta_0\}$ there is a function $\tau(\cdot) \in \mathrm{Hom}$ such that

$$\lim_{t \to +\infty} |x(t, x_0) - x(\tau(t), y_0)| = 0$$

then the solution $x(t, x_0)$ is said to be asymptotically Zhukovskii stable.

In other words, Zhukovskii stability is Lyapunov stability for a suitable reparametrization of each of the perturbed trajectories.

We recall that, by definition, instability in the sense of Lyapunov (Poincaré, Zhukovskii) is simply the negation of the relevant notion of stability.

Obviously, Lyapunov stability implies Zhukovskii stability, and Zhukovskii stability implies Poincaré stability.

These definitions usually assume that all solutions are defined for all $t \in [0, +\infty)$.

We recall that, for equilibrium states, all three definitions are equivalent. For periodic solutions, it is easy to show that Poincaré stability and Zhukovskii stability are equivalent [14]. In non-linear systems one often encounters the situation in which a periodic solution is asymptotically stable in Zhukovskii's sense (hence also in Poincaré's sense) but unstable in Lyapunov's sense [14]. In that case one introduces the notion of an asymptotic phase $c(y_0)$ which, in the context of the definition of asymptotic Zhukovskii stability, corresponds to a reparametrization $\tau(t) = t + c(y_0)$. This alone shows that Lyapunov instability cannot be characterized by the property of trajectories to "repel" one another on such intrinsically unstable objects as strange attractors.

Poincaré instability is also not characterizable by "repulsion" for strange attractors, but for a different reason. Numerical experiments frequently reveal the situation (for example, on the Lorentz attractor [5]) in which at least one semi-trajectory $x(t, x_0)$ densely fills the attractor. But then inequality (2) holds for any $\varepsilon > 0$ and any point y_0 of the attractor. Consequently, all such trajectories $x(t, x_0)$ are Poincaré stable.

However, experiments on strange attractors may reveal mutual "repulsion" of trajectories with passing time. Such "repulsion" corresponds to Zhukovskii instability. Of the three concepts defined above, therefore, Zhukovskii instability most adequately describes the behaviour of trajectories on strange attractors.

Various examples demonstrating Zhukovskii unstable flows on two-dimensional compact manifolds may be found in [14].

When studying Zhukovskii stability and instability, it is important that non-classical linearizations arise here in a natural manner. To describe such linearizations, let us assume that all the semi-trajectories $x(t, x_0)$, $t \geq 0$ of system (1) under consideration lie in a certain compact set G and $f(x) \neq 0$, $\forall x \in G$. In that case one has the following result, establishing the existence of a special reparametrization of the perturbed trajectories.

Lemma 1 [14]. For any number $T > 0$, a number $\delta(T) > 0$ exists such that, for any vector y_0 in the set $\{y \mid |x_0 - y| \leq \delta(T), (y - x_0)^* f(x_0) = 0\}$, a differentiable function $\tau(\cdot) \in \text{Hom}$ exists which satisfies the relation

$$(x(t, x_0) - x(\tau(t), y_0))^* f(x(t, x_0)) = 0, \quad \forall t \in [0, T] \tag{3}$$

Under these conditions

$$\frac{d\tau(t)}{dt} = 1 - \frac{f(x(t, x_0))^*}{|f(x(t, x_0))|^2} \left(\frac{\partial f}{\partial x}(x(t, x_0)) + \left(\frac{\partial f}{\partial x}(x(t, x_0)) \right)^* \right) \times$$
$$\times (x(t, x_0) - x(\tau(t), y_0)) + O(|x(t, x_0) - x(\tau(t), y_0)|^2) \tag{4}$$

where $(\partial f/\partial x)(x(t, x_0))$ is the value of the Jacobian of the vector function f at the point $x(t, x_0)$; the symbol $O(\upsilon)$ denotes a quantity such that, for sufficiently small υ, we have $|O(\upsilon)| \leq c\upsilon$, where c is a certain number and the asterisk denotes transposition.

It follows from (4) that, for the reparametrization $\tau(t)$ indicated in the lemma, the linear system of the first approximation along the trajectory $x(t, x_0)$ has the form

$$\frac{dz}{dt} = \left[\frac{\partial f}{\partial x}(x(t, x_0)) - g(x(t, x_0)) \left(\frac{\partial f}{\partial x}(x(t, x_0)) + \left(\frac{\partial f}{\partial x}(x(t, x_0)) \right)^* \right) \right] z$$
$$f(x(t, x_0))^* z = 0, \quad g(x(t, x_0)) = \frac{f(x(t, x_0))\, f(x(t, x_0))^*}{|f(x(t, x_0))|^2} \tag{5}$$

We will also consider the classical linearization

$$\frac{dy}{dt} = \frac{\partial f}{\partial x}(x(t, x_0))\, y \tag{6}$$

Lemma 2. If $y(t)$ is a solution of system (6), then

$$z(t) = (I - g(x(t, x_0)))\, y(t)$$

is a solution of system (5).

This can be verified by direct substitution of $z(t)$ into the right- and left-hand sides of system (5).

We now recall the definition of the Lyapunov exponents of a linear system

$$dx/dt = A(t)\, x, \quad x \in \mathbf{R}^n \tag{7}$$

where $A(t)$ is a continuous $n \times n$ matrix. To that end, we let $X(t)$ denote the fundamental matrix of system (7) with initial data $X(0) = I$, where I is the identity matrix. Let $\rho_i(t)$ denote the square roots of the eigenvalues of the matrix $X(t)^* X(t)$; they are known as the singular numbers of the matrix $X(t)$. Henceforth we will assume that $\rho_1(t) \geq \rho_2(t) \geq \ldots \geq \rho_n(t)$.

It is an almost obvious, but important, fact that the singular numbers of $X(t)$ have a very simple geometrical interpretation. The operator $X(t)$ maps the unit sphere of the space $\mathbf{R}^n$ into an ellipsoid whose principal semi-axes coincide with the singular numbers $\rho_i(t)$. Thus, the singular numbers characterize the compressing and expanding properties of the operator $\mathbf{X}(t)$: $\mathbf{R}^n \to \mathbf{R}^n$. In particular, we have an estimate

$$\rho_n(t)\, |x| \leq |X(t)\, x| \leq \rho_1(t)\, |x|, \quad \forall x \in \mathbf{R}^n$$

These considerations imply the following simple proposition.

Lemma 3. The singular numbers $\alpha_i(t)$ of the matrix $X(t)S$ satisfy the estimate $\varkappa_1\rho_i(t) \leqslant \alpha_i(t) \leqslant \varkappa_2\rho_i(t)$, where κ_1 and κ_2 are, respectively, the minimum and maximum singular numbers of the $n \times n$ matrix S.

Definition 4. The Lyapunov exponent ν_j of system (7) is defined as the number

$$\nu_j = \ln \overline{\lim_{t\to+\infty}} \rho_j(t)^{1/t} \tag{8}$$

It follows from Lemma 3 that the condition $X(0) = 1$ in the definition of Lyapunov exponents may be omitted, since the above definition yields the same numbers ν_j for the matrices $X(t)S$ and $X(t)$, $\det S \neq 0$.

We will now investigate the Lyapunov and Zhukovskii stability of solutions $x(t, t_0)$ of system (1), on the assumption that $x_0 \in \Omega$, where Ω is some bounded open subset of $\mathbf{R}^n$.

As to Lyapunov stability, we shall consider the more general case of a system

$$dx/dt = f(t, x), \qquad x \in \mathbf{R}^n \tag{9}$$

where $f(t,x)$ is a twice continuously differentiable vector function. Consider the solution $x(t,x_0)$ of system (9) with initial data $x(0,x_0) = x_0$. Let us assume that the solutions of system (9) $x(t,x_0)$, $x_0 \in \Omega$ and the singular numbers $\alpha_1(t,x_0) \geqslant \alpha_2(t,x_0) \geqslant \ldots \geqslant \alpha_n(t,x_0)$ of the fundamental matrix of the system

$$dy/dt = \partial f(t,x)/\partial x|_{x=x(t,x_0)}y \tag{10}$$

together with some continuous function $\alpha(t)$, satisfy the relationship

$$\alpha_1(t,x_0) \leqslant \alpha(t), \quad \forall t \geqslant 0, \quad \forall x_0 \in \Omega \tag{11}$$

Theorem 1. Let the function $\alpha(t)$ be bounded in the interval $(0, +\infty)$. Then the solution $x(t, x_0)$, $x_0 \in \Omega$ is Lyapunov stable. If, moreover

$$\lim_{t\to+\infty} \alpha(t) = 0$$

then the solution $x(t,x_0) \in \Omega$ is asymptotically Lyapunov stable.

Proof. Put $F_t y = x(t, y)$, $x(0, y) = y$, that is, $F_t y$ is the translation operator along solutions of system (9): $(F_t y)' = X(t, y)$, $X(0, y) = I$, where $(F_t y)'$ is the derivative with respect to y of the operator $F_t y$, which is identical with the fundamental matrix of the linear system (10).

It is well known [15] that, under our assumptions, for any vectors y, z and number $t \geqslant 0$, a vector w exists such that

$$|w-y| \leqslant |y-z|, \quad |F_t y - F_t z| \leqslant |(F_t w)'|\,|y-z| \tag{12}$$

Hence, using (11), we deduce that for any y_0 such that

$$\{w \mid |w-x_0| \leqslant |x_0-y_0|\} \subset \Omega$$

we have an estimate

$$|F_t x_0 - F_t y_0| \leqslant |x_0 - y_0| \sup \alpha_1(t,w) \leqslant \alpha(t)|x_0 - y_0|, \quad \forall t \geqslant 0 \tag{13}$$

where the supremum is evaluated over the set

$$w \in \{w \mid |w-x_0| \leqslant |x_0-y_0|\} \tag{14}$$

Estimate (13) immediately implies the conclusion of Theorem 1.

Let us assume now that, instead of inequality (11), the fundamental matrix $X(t, x_0)$ of system (10) with initial condition $X(0, x_0) = I$, together with some vector function $\xi(t)$, satisfy the relations

$$|\xi(t)| = 1, \quad \forall t \geqslant 0; \quad \max_i \inf_\Omega |X_i(t,x_0)\xi(t)| \geqslant \alpha(t) \tag{15}$$

where $X_i(t, x_0)$ is the ith row of the matrix $X(t, x_0)$.

Theorem 2. Suppose the function $\alpha(t)$ satisfies the relation

$$\varliminf_{t \to +\infty} \alpha(t) = +\infty \tag{16}$$

Then the solution $x(t, x_0)$, $x_0 \in \Omega$ is Lyapunov unstable.

Proof. Having fixed a pair $x_0 \in \Omega$ and $t > 0$, we choose, in any δ-neighbourhood of x_0, a vector y_0 such that

$$x_0 - y_0 = \delta\, \xi(t)$$

We will assume that δ is so small that $\{w \mid |w - x_0| \leqslant \delta\} \subset \Omega$.

Let $F_{it}z = x_i(t, z)$ denote the ith component of the vector function $x(t, z)$.

It is well known [15] that, for any fixed numbers t, i and vectors x_0, y_0 a vector $w_i \in \mathbf{R}^n$ exists such that

$$|x_0 - w_i| \leqslant |x_0 - y_0|, \quad F_{it}x_0 - F_{it}y_0 = X_i(t, w_i)(x_0 - y_0) \tag{17}$$

Using this relation, we obtain an estimate

$$|F_t x_0 - F_t y_0| = \left(\sum_i |X_i(t, w_i)(x_0 - y_0)|^2 \right)^{1/2} \geqslant$$

$$\geqslant \delta \max\{|X_1(t, w_1)\xi(t)|, \ldots, |X_n(t, w_n)\xi(t)|\} \geqslant \delta \max_i \inf |X_i(t, w)\xi(t)| \geqslant \delta\alpha(t)$$

where the infinum is evaluated over the set (14).

It follows from this estimate and from condition (16) that, for any positive numbers ε and δ, a vector y_0 and a number t exist such that

$$|x_0 - y_0| \leqslant \delta, \quad |F_t x_0 - F_t y_0| \geqslant \varepsilon$$

This means that the solution $x(t, x_0)$ is Lyapunov unstable.

Theorem 1 establishes the asymptotic stability, in Lyapunov's sense, of a flow of solutions with initial data in Ω if their linearizations have negative Lyapunov exponents. One does not encounter here the effects that Perron found in non-regular linearizations of individual solutions [2]. Roughly speaking, in this setting the condition that the singular numbers of the fundamental matrix of the linearization decrease uniformly and exponentially "as functions of t_0" (and this is one of the properties of regular stable linear systems) in the classical theorems of Lyapunov and his successors [1, 16] is replaced by the condition that the decrease be uniform "as a function of x_0". Thus, Perron effects are possible only at the boundaries of a flow which is stable in the first approximation.

Conditions (15) and (16) in Theorem 2 essentially express the requirement that at least one of the Lyapunov exponents of linearizations of a flow of solutions with initial data in Ω be positive, with a slight addition: the "unstable directions $\xi(t)$" (or unstable manifolds) of these solutions depend, coordinate by coordinate, on the initial data. In fact, if this property holds, then, if necessary, considering the domain Ω as the union of domains Ω_i of arbitrarily small diameter in which conditions (15) and (16) hold, we obtain Lyapunov instability of the whole flow of solutions with initial data in Ω.

Let us now consider system (1) and the linear approximations (5) for solutions $x(t, x_0)$, $x_0 \in \Omega$, assuming that $x(t, x_0) \in \Omega$, $\forall t \geqslant 0$, and $f(x) \neq 0$, $\forall x \in \Omega$. It is well known that $f(x(t, x_0))$ is a solution of system (6). We may therefore consider a linearly independent system of solutions $f(x(t, x_0)), y_2(t), \ldots, y_n(t)$ of system (6) and matrices

$$Y(t, x_0) = (f(x(t, x_0)), \tilde{Y}(t)), \quad \tilde{Y}(t, x_0) = (y_2(t), \ldots, y_n(t))$$

$$Z(t, x_0) = (I - g(x(t, x_0)))\, Y(t)$$

Let us assume that the singular numbers $\beta_1(t, x_0) \geqslant \ldots \geqslant \beta_n(t, x_0)$ of the fundamental matrix $Z(t, x_0)$ of system (5) and some continuous function $\beta(t)$ satisfy the estimate

$$\beta_1(t, x_0) \leqslant \beta(t), \qquad \forall t \geqslant 0 \quad \forall x_0 \in \Omega \tag{18}$$

Theorem 3. If

$$\lim_{t\to+\infty} \beta(t) = 0$$

then the solution $x(t, x_0)$, $x_0 \in \Omega$, is asymptotically Zhukovskii stable.

Proof. Fix a number T such that $\beta(t) \leq 1/2$, $\forall t \geq T$. By Lemma 1, we can choose $\delta(T) > 0$ such that, for $x_0 \in \Omega$ and any y_0 satisfying the inequalities

$$|y_0 - x_0| \leq \delta(T), \quad (y_0 - x_0)^* f(x_0) = 0$$

a differentiable function $\tau(t)$ satisfying (3) and (4) exists.

Choose $\delta(T)$ also with this property and such that

$$\{y \mid |x_0 - y| \leq \delta(T)\} \subset \Omega$$

We now make the change of independent variable $x(t) \to x(\tau(t))$ in system (1)

$$\frac{dx}{dt} = \frac{dx}{d\tau}\frac{d\tau}{dt} = f(x(\tau(t)))\frac{d\tau(t)}{dt} \tag{19}$$

Clearly, this system admits of a certain analogue of the first integral (3), and by (4) the linearization of system (19) along the solution $x(t, x_0)$ has the form (5). Therefore, repeating the arguments adduced in the proof of Theorem 1 and using the obvious equality $x(\tau(t), x_0) = x(t, x_0)$, we obtain

$$|x(T, x_0) - x(\tau(T), y_0)| \leq \beta(T)|x_0 - y_0| \leq |x_0 - y_0|/2 \tag{20}$$

Now considering the vector $q_0 = x(T, x_0)$ as new initial data and again applying Theorem 1, we repeat all the above arguments in the interval $[0, T]$ for the solution $x(t, q_0)$. Then, in view of (20), we obtain

$$|x(2T, x_0) - x(\tau(2T), y_0)| \leq |x_0 - y_0|/4$$

Continuing in this way, we get

$$|x(t, x_0) - x(\tau(t), y_0)| \leq \max_{t \geq 0} \beta(t)|x_0 - y_0|$$

$$\lim_{t\to+\infty} |x(t, x_0) - x(\tau(t), y_0)| = 0$$

These relationships mean that the solution $x(t, x_0)$ is asymptotically Zhukovskii stable.

We now consider a point $x_0 \in \Omega$ and a set $\Phi(x_0) = \{u \mid u \in \Omega, u^* f(x_0) = 0, |u - x_0| \leq \gamma\}$ (where γ is some number).

Let us assume that, instead of inequality (18), the previously introduced matrix $Z(t, u)$ and some vector function $\xi(t)$ satisfy the relations

$$\xi(t)^* f(x_0) = 0, \quad |\xi(t)| = 1, \quad \forall t \geq 0$$

$$\max_i \inf_{\Phi(x_0)} |Z_i(t, u)\xi(t)| \geq \alpha(t), \quad \forall t \geq 0 \tag{21}$$

Theorem 4. Suppose the function $\alpha(t)$ satisfies (16). Then the solution $x(t, x_0)$ is Zhukovskii unstable.

Proof. Suppose the contrary, that is, the solution $x(t, x_0)$ is Zhukovskii stable. Then a number $\delta > 0$ exists such that, for any y_0 satisfying the relations $|x_0 - y_0| \leq \delta$ and $(x_0 - y_0)^* f(x_0) = 0$, a parametrization $\tau(t)$ exists for which (3) and (4) hold for all $t \geq 0$ [14]. Thus, one can change to system (19) for all $t \geq 0$. Repeating the arguments used in the proof of Theorem 2 and noting that the linearization of system (19) along the solution $x(t, x_0)$ is system (5), we conclude that for any $t > 0$ a vector y_0 exists such that

$$|x_0 - y_0| \leq \delta, \quad (x_0 - y_0)^* f(x_0) = 0, \quad |x(t, x_0) - x(\tau(t), y_0)| \geq \alpha(t)\delta$$

Hence, by (3) and condition (16), the solution $x(t, x_0)$ must be Zhukovskii unstable.

We observe that the main difference between Theorems 3 and 4 and the previous results of [13, 14] is that the regularity assumption for linearizations has been dropped. Instead, it is assumed that the stability (or instability) property of the first approximation is preserved for small changes in the initial data x_0.

REFERENCES

1. LYAPUNOV, A. M., *The General Problem of the Stability of Motion.* Gostekhizdat, Moscow, 1950.
2. PERRON, O., Die Stabilitätsfrage bei Differentialgleichungen. *Math. Zeitschrift*, 1930, **32,** 5, 703–728.
3. CHETAYEV, N. G., A theorem on instability for regular systems. *Prikl. Mat. Mekh.*, 1944, **8,** 4, 323–326.
4. CHETAYEV, N. G., *The Stability of Motion.* Gostekhizdat, Moscow, 1946.
5. SINAI, Ya. G. and SHIL'NIKOV, L. P. (Eds), *Strange Attractors.* Mir, Moscow, 1981.
6. RABINOVICH, M. I., Stochastic self-excited oscillations and turbulence. *Uspekhi Fiz. Nauk*, 1978, **125,** 1, 123–168.
7. REITMANN, V., *Reguläre und chaotische Dynamik.* Teubner, Stuttgart, 1996.
8. SCHUSTER, H. G., *Deterministic Chaos: An Introduction*, 3rd edition. John Wiley, New York, 1995.
9. NEIMARK, Yu. I. and LANDA, P. S., *Stochastic and Chaotic Oscillations.* Nauka, Moscow, 1987.
10. ANISHCHENKO, V. S., *Complicated Oscillations in Simple Systems.* Nauka, Gl. Red. Fiz.-Mat. Lit., Moscow, 1990.
11. ANDRONOV, A. A. and VITT, A. A., On stability in Lyapunov's sense. *Zh. Eksper. Teor. Fiz.*, 1933, **3,** 5, 373–374.
12. DEMIDOVICH, B. P., *Lectures on the Mathematical Theory of Stability.* Nauka, Moscow, 1967.
13. DEMIDOVICH, B. P., The orbital stability of bounded solutions of an autonomous system I, II. *Differents. Uravn.*, 1968, **4,** 4, 575–588; **4,** 8, 1359–1373.
14. LEONOV, G. A., PONOMARENKO, D. V. and SMIRNOVA, V. B., Local instability and localization of attractors. From stochastic generator to Chua's systems. *Acta Appl. Math.*, 1995, **40,** 179–243.
15. ZORICH, V. A., *Mathematical Analysis*, Pts I and II. Nauka, Moscow, 1981–1984.
16. MALKIN, I. G., *Theory of the Stability of Motion.* Nauka, Moscow, 1966.

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